

# FORMATION OF SHOCKS IN HIGHER-ORDER NONLINEAR DISPERSION PDES: NONUNIQUENESS AND NONEXISTENCE OF ENTROPY

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ABSTRACT. Formation of shocks for the third-order nonlinear dispersion PDE

$$(0.1) \quad u_t = (uu_x)_{xx} \quad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

is studied by construction various self-similar solutions exhibiting gradient blow-up in finite time, as  $t \rightarrow T^- \in (0, \infty)$ , with locally bounded final time profiles  $u(x, T^-)$ . These are shown to admit infinitely many discontinuous shock-type similarity extensions for  $t > T$ , all of them satisfying generalized Rankine–Hugoniot’s condition at shocks. As a result, the principal nonuniqueness of solutions of the Cauchy problem after blow-up time is inherited, and, under certain hypothesis, any sufficiently general “entropy-like” approach to such problems to detect a unique continuation beyond singularity becomes illusive. In other words, any attempt to create “entropy theory” for (0.1) along the lines of the classic one for scalar conservation laws (developed by Oleinik and Kruzhkov in the 1950-60s), such as Euler’s equation  $u_t + uu_x = 0$  in  $\mathbb{R} \times \mathbb{R}_+$ , or by using other more recent successful ideas for hyperbolic systems, is hopeless, regardless the fact that these PDEs have a number of similar features of formation of shock and rarefaction waves.

## 1. INTRODUCTION: NONLINEAR DISPERSION PDES AND MAIN DIRECTIONS OF STUDY

We study “micro-structure” of shock-type finite time singularities that can occur in higher-order nonlinear dispersion equations (NDEs). To describe key features of formation of such single point singularities, it suffices to consider the NDE of the minimal third order. Its study eventually leads to quite a pessimistic conclusion concerning uniqueness of “entropy” solutions and on nonexistence of any hypothetical entropy.

**1.1. NDEs: nonlinear dispersion equations.** In the present paper, we continue our study began in [5]–[8] of basic aspects of singularity formation and approaches to existence-uniqueness-entropy for odd-order *nonlinear dispersion* (or *dispersive*) equations. The simplest canonical model is the *third-order quadratic NDE* (the NDE-3)

$$(1.1) \quad u_t = (uu_x)_{xx} \equiv uu_{xxx} + 3u_xu_{xx} \quad \text{in } \mathbb{R} \times (0, T), \quad T > 0.$$

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We pose for (1.1) the Cauchy problem (the CP) with locally bounded and integrable initial data

$$(1.2) \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}.$$

It is principal that we consider the CP, where the solution  $u(x, t)$  is supposed to be defined by initial data (1.2) only, plus, of course a generalized Rankine–Hugoniot-type condition on the speed of propagation of shocks, which follows from the equation integrated in a shock neighbourhood. In other words, we are assuming that the CP does not require any *a priori* posed conditions on the shock wave lines (though of course such ones exist and can be determined *a posteriori*). Otherwise, with such conditions, we arrive at a *free boundary problem* (an FBP) for the NDE (1.1), which requires other mathematical methods of study and can be well-posed (unlike the CP). We will touch possible FBP settings for (1.1).

The physical motivation of the NDEs such as (1.1) and other odd-order nonlinear PDEs, which appear in many areas of application, with a large number of key references, are available in surveys in [7, § 1] or [8, § 1], so we omit any discussion on these applied issues.

**1.2. NDEs in general PDE theory.** For future convenience, we briefly present a more mathematical discussion around NDEs; cf. [8, § 1]. In the framework of general theory of nonlinear evolution PDEs of the first order in time, the NDE (1.1) appears the third among other canonical evolution quasilinear degenerate equations:

$$(1.3) \quad u_t = -\frac{1}{2}(u^2)_x \quad (\text{the conservation law}),$$

$$(1.4) \quad u_t = \frac{1}{2}(u^2)_{xx} \quad (\text{the porous medium equation}),$$

$$(1.5) \quad u_t = \frac{1}{2}(u^2)_{xxx} \quad (\text{the nonlinear dispersion equation}),$$

$$(1.6) \quad u_t = -\frac{1}{2}(|u|u)_{xxxx} \quad (\text{the 4}^{\text{th}}\text{-order nonlinear diffusion equation}).$$

In (1.6), the quadratic nonlinearity  $u^2$  is replaced by the monotone one  $|u|u$  in order to keep the parabolicity on solutions of changing sign. The same can be done in the PME (1.4), though this classic version is parabolic on nonnegative solutions, a property that is preserved by the Maximum Principle. Further extensions of the list by including

$$(1.7) \quad u_t = -\frac{1}{2}(u^2)_{xxxxx} \quad (\text{the NDE-5}) \quad \text{and}$$

$$(1.8) \quad u_t = \frac{1}{2}(|u|u)_{xxxxx} \quad (\text{the 6}^{\text{th}}\text{-order nonlinear diffusion equation}), \text{ etc.,}$$

are not that essential. These PDEs belong to the same families as (1.5) and (1.6) respectively with similar covering mathematical concepts (but more difficult in some details).

Mathematical theory of the first two equations, (1.3) (see detailed survey and references below) and (1.4) (for quoting main achievements of PME theory developed in the 1950–80s, see e.g., [9, Ch. 2]), was essentially completed in the twentieth century. It is curious that looking more difficult the fourth-order nonlinear diffusion equation (1.6) has a monotone operator in  $H^{-2}$ , so the Cauchy problem admits a unique weak solution as

follows from classic theory of monotone operators; see Lions [13, Ch. 2]. Of course, some other qualitative properties of solutions of (1.6) are more difficult and remain open still.

It turns out that, rather surprisingly, the third order NDE (1.5) is the only one in this basic list that has rather obscure understanding and lacking of a reliable mathematical basis concerning generic singularities, shocks, rarefaction waves, and entropy-like theory.

**1.3. Mathematical preliminaries: analogies with conservation laws, Riemann's problems, and earlier results.** As a key feature of our analysis, equation (1.1) inherits clear similarities of the behaviour for the first-order conservation laws such as 1D *Euler's equation* (same as (1.3)) from gas dynamics

$$(1.9) \quad u_t + uu_x = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

whose entropy theory was created by Oleinik [14, 15] and Kruzhkov [12] (equations in  $\mathbb{R}^N$ ) in the 1950–60s; see details on the history, main results, and modern developments in the well-known monographs [1, 3, 19]<sup>1</sup>.

As for (1.9), in view of the full divergence of the equation (1.1), it is natural to define weak solutions. For convenience, we present here a standard definition, mentioning however that, in fact, the concept of weak solutions for NDEs even in fully divergent form is not entirely consistent and/or very helpful, to say nothing about other non-divergent equations admitting no standard weak formulation at all; see [8].

**Definition 1.1.** *A function  $u = u(x, t)$  is a weak solution of (1.1), (1.2) if*

- (i)  $u^2 \in L^1_{\text{loc}}(\mathbb{R} \times (0, T))$ ,
- (ii)  $u$  satisfies (1.1) in the weak sense: for any test function  $\varphi(x, t) \in C_0^\infty(\mathbb{R} \times (0, T))$ ,

$$(1.10) \quad \iint u \varphi_t = \frac{1}{2} \iint u^2 \varphi_{xx},$$

and (iii) satisfies the initial condition (1.2) in the sense of distributions,

$$(1.11) \quad \text{ess} \lim_{t \rightarrow 0} \int u(x, t) \psi(x) = \int u_0(x) \psi(x) \quad \text{for any} \quad \psi \in C_0^\infty(\mathbb{R}).$$

The assumption  $T < \infty$  is often essential, since, unlike (1.9), the NDE (1.1) can produce complete blow-up from bounded data, [8, § 4]. Thus, again similar to (1.9), one observes a typical difficulty: according to Definition 1.1, both discontinuous step-like functions

$$(1.12) \quad S_{\mp}(x) = \mp \text{sign } x = \mp \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0, \end{cases}$$

are weak *stationary* solutions of (1.1) satisfying

$$(1.13) \quad (u^2)_{xxx} = 0$$

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<sup>1</sup>First study of discontinuous shocks for quasilinear equations was performed by Riemann in 1858 [18] (by Riemann's method); see [2, 17] for details. The implicit solution of the problem (1.9),  $u = u_0(x - ut)$  (containing the key wave “overtaking” effect), was obtained earlier by Poisson in 1808 [16]; see [17].

in the weak sense, since in (1.10)  $u^2(x) = S_{\mp}^2(x) \equiv 1$  is  $C^3$  smooth ( $x = 0$  does not count). Again referring to entropy theory for conservation laws (1.9), it is well-known that

$$(1.14) \quad \begin{aligned} u_-(x, t) &\equiv S_-(x) \text{ is an entropy shock wave, and} \\ u_+(x, t) &\equiv S_+(x) \text{ is not an entropy solution.} \end{aligned}$$

This means that

$$(1.15) \quad u_-(x, t) \equiv S_-(x) = -\text{sign } x$$

is the unique entropy solution of the PDE (1.9) with the same initial data  $S_-(x)$ . On the contrary, taking  $S_+$  initial data yields the *rarefaction wave* with a simple similarity piece-wise linear structure

$$(1.16) \quad u_0(x) = S_+(x) = \text{sign } x \implies u_+(x, t) = F\left(\frac{x}{t}\right) = \begin{cases} -1 & \text{for } x < -t, \\ \frac{x}{t} & \text{for } |x| < t, \\ 1 & \text{for } x > t. \end{cases}$$

The questions on justifying the same classification of main two Riemann's problems with data (1.12) for the NDE (1.1) and to construct the corresponding rarefaction wave for  $S_+(x)$ , as an analogy of (1.16) for the conservation law were addressed in [8, 6]. This was done by studying the following self-similar solutions of (1.1):

$$(1.17) \quad u_-(x, t) = f(y), \quad y = \frac{x}{(-t)^{1/3}} \implies (ff')'' = \frac{1}{3} f' y \quad \text{in } \mathbb{R}, \quad f(\mp\infty) = \pm 1.$$

Here, by translation, the blow-up time in (1.17) reduces to  $T = 0$ . It was shown that, in the sense of distributions or in  $L_{\text{loc}}^1$ ,

$$(1.18) \quad u_-(x, t) \rightarrow S_-(x) \quad \text{as } t \rightarrow 0^-.$$

Therefore,  $S_-(x)$  is a  $\delta$ -entropy shock wave (see concepts in [6]), while  $S_+(x)$  is not and creates a typical *rarefaction wave* given by the global similarity solution

$$(1.19) \quad u_+(x, t) = F(y), \quad y = \frac{x}{t^{1/3}} \implies (FF')'' = -\frac{1}{3} F' y \quad \text{in } \mathbb{R}, \quad F(\mp\infty) = \mp 1,$$

where indeed  $F(y) \equiv -f(y)$ , so that, in the same sense as in (1.18),

$$u_+(x, t) \rightarrow S_+(x) \quad \text{as } t \rightarrow 0^+.$$

**1.4. Layout of the paper: new shock patterns and nonuniqueness.** In Section 2, we study new shock patterns, which are induced by other similarity solutions:

$$(1.20) \quad u_-(x, t) = (-t)^\alpha f(y), \quad y = \frac{x}{(-t)^\beta}, \quad \beta = \frac{1+\alpha}{3}, \quad \text{where } \alpha \in (0, \frac{1}{2}) \text{ and}$$

$$(1.21) \quad (ff')'' - \beta f' y + \alpha f = 0 \quad \text{in } \mathbb{R}_-, \quad f(0) = f''(0) = 0.$$

The anti-symmetry conditions in (1.21) allow to extend the solution to the positive semi-axis  $\{y > 0\}$  by  $-f(-y)$  to get a global pattern. The case  $\alpha < 0$  in (1.20), corresponding to the strong complete blow-up was studied in detail in [8, § 4] in the parameter range

$$(1.22) \quad \alpha \in \left[-\frac{1}{10}, 0\right), \quad \text{where } a_c = -\frac{1}{10} \text{ is a special critical exponent.}$$

For convenience, we revise some of these blow-up results in the range (1.22) in Section 2.

Obviously, the solutions (1.17), which are suitable for Riemann's problems, correspond to the simple case  $\alpha = 0$  in (1.20). We prove that, using positive  $\alpha$ , allow to get first *gradient blow-up* at  $x = 0$  as  $t \rightarrow 0^-$ , as a weak discontinuity, where the final time profile remains locally bounded and continuous:

$$(1.23) \quad u_-(x, 0^-) = \begin{cases} C_0|x|^{\frac{\alpha}{\beta}} & \text{for } x < 0, \\ -C_0|x|^{\frac{\alpha}{\beta}} & \text{for } x > 0, \end{cases}$$

where  $C_0 > 0$  is an arbitrary constant. Note that  $\frac{\alpha}{\beta} < 1$  for  $\alpha < \frac{1}{2}$ .

Therefore, the wave braking ("overturning") begins at  $t = 0$ , and, in Section 3, we show that it is performed again in a self-similar manner and is described by similarity solutions

$$(1.24) \quad u_+(x, t) = t^\alpha F(y), \quad y = \frac{x}{t^\beta}, \quad \beta = \frac{1+\alpha}{3}, \quad \text{where}$$

$$(1.25) \quad \begin{cases} (FF')'' + \beta F'y - \alpha F = 0 & \text{in } \mathbb{R}_-, \\ F(0) = F_0 > 0, \quad F(y) = C_0|y|^{\frac{\alpha}{\beta}}(1 + o(1)) & \text{as } y \rightarrow -\infty, \end{cases}$$

where the constant  $C_0 > 0$  is fixed by blow-up data (1.23). The asymptotic behaviour as  $y \rightarrow -\infty$  in (1.25) guarantees the continuity of the global discontinuous pattern (with  $F(-y) \equiv -F(y)$ ) at the singularity blow-up instant  $t = 0$ , so that

$$(1.26) \quad u_-(x, 0^-) = u_+(x, 0^+) \quad \text{in } \mathbb{R}.$$

Rather surprisingly, for a fixed  $C_0 > 0$  in (1.23) obtained by blow-up evolution as  $t \rightarrow 0^-$ , we find infinitely many solutions of the extending problem (1.25). This family of solutions  $\Phi_{C_0} = \{F(y; F_0), F_0 > 0\}$  is a one-dimensional curve parameterized by arbitrary

$$(1.27) \quad F_0 = F(0) > 0,$$

which does not have any clear boundary or extremal points. We also show that for  $F_0 = 0$  (1.25) does not have a solution. In other words, this solution family does not contain any "minimal", "maximal", or "extremal" points in any reasonable sense, which might play a role of a unique "entropy" one chosen by introducing a hypothetical entropy inequalities, conditions, or otherwise.

A first immediate consequence of our similarity blow-up/extension analysis is as follows:

$$(1.28) \quad \boxed{\text{in the CP, formation of shocks for the NDE (1.1) leads to nonuniqueness.}}$$

The second conclusion is more subtle and is based on the mentioned above fact on the homogeneous structure of the functional set  $\Phi_{C_0}$ : if  $\Psi_{C_0} = \{u_+(x, t), F \in \Phi_{C_0}\}$  is the whole set of weak solutions of (1.1) with initial data (1.23), then, for the Cauchy problem for (1.1),

$$(1.29) \quad \boxed{\text{there exists no general "entropy mechanism" to choose a unique solution.}}$$

Of course, we cannot exclude a hypothetical situation, when there exists another, non-similarity solution of this problem, which thus does not belong to the family  $\Psi_{C_0} = \{u_+\}$  and is the right solution with a proper entropy-type specification. In our opinion, this is suspicious, so that we claim that (1.28) and (1.29) show that the problem of uniqueness

of weak solutions for the NDEs such as (1.1) cannot be solved in principal. On the other hand, in a FBP setting by adding an extra suitable condition on shock lines, the problem might be well-posed with a unique solution, though proofs can be very difficult.

In other words, *non-uniqueness* in the CP is a non-removable issue of PDE theory for higher-order degenerate nonlinear odd-order equations (and possibly not only those). In fact, the non-uniqueness of solutions of (1.25) has a pure and elementary dimensional nature. Indeed, this is a third-order ODE with the general solution depending on three parameters, which are more than and excessively enough to shoot the right behaviour at infinity and hence to get the continuity (1.26) at the blow-up time. This guarantees non-uniqueness of the extension for  $t > 0$  and is a core of this difficulty.

Increasing the order of the PDE under consideration, we then enlarge the dimension of the parametric space, and this surely will imply even stronger non-uniqueness conclusions. For example, non-uniqueness and non-entropy features are available for the *fifth-order nonlinear dispersion equation* (the NDE-5)

$$(1.30) \quad u_t = -(uu_x)_{xxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+$$

and others; see basic models in [6, 7].

## 2. GRADIENT BLOW-UP SIMILARITY SOLUTIONS

In this section, we consider the blow-up ODE problem (1.21). Actually, this ODE is not that difficult for application of standard shooting methods, which in greater detail are explained in [8, § 3,4]. Moreover, even similar fifth-order ODEs associated with the shocks for the NDE-5 (1.30) also admit similar shooting analysis [7], though, in view of the essential growth of the dimension of the phase space, some more delicate issues on, say, uniqueness of certain orbits, become very difficult or even remain open. Therefore, in what follows, we will omit or even will not mention some technical details concerning the problem (1.21) and recommend consulting [8] in case of troubles. We will widely use numerical methods for illustrating and even justifying some of our conclusions. For the third-order equations such as (1.1), this and further numerical constructions are performed by the **MatLab** with the standard **ode45** solver therein; see more details in [8, § 3,4].

Let us begin with a simpler fact concerning this problem and some simple asymptotics for matching purposes. We recall the elementary symmetry of the ODE (1.21)

$$(2.1) \quad \begin{cases} f \mapsto -f, \\ y \mapsto -y, \end{cases}$$

which allows us to put two conditions at the origin. Such solutions have a sufficiently regular asymptotic expansion near the origin: for any  $A < 0$ , there exists a unique solution of the ODE (1.21), satisfying

$$(2.2) \quad f(y) = Ay + \frac{1-2\alpha}{72} y^3 + \frac{(1-2\alpha)^2}{72^2} \frac{1}{A} y^5 + \dots \quad \text{as} \quad y \rightarrow 0.$$

The uniqueness of such asymptotics is traced out by using Banach's Contraction Principle applied to the equivalent integral equation in the metric of  $C(-\delta, \delta)$ , with  $\delta > 0$  small.

We use the following scaling invariance of the ODE in (1.21): if  $f_1(y)$  is a solution, then

$$(2.3) \quad f_a(y) = a^3 f_1\left(\frac{y}{a}\right) \quad \text{is a solution for any } a \neq 0.$$

By scaling (2.3), the parameter  $A < 0$  can be reduced to a single value, say  $A = -1$ .

Let us describe the necessary bundles of the 3D (in fact, 4D, for the non-autonomous case) dynamical system (1.21). Firstly, due to the scaling symmetry (2.3), there exists the explicit solution

$$(2.4) \quad f_*(y) = \frac{1}{60} y^3 < 0 \quad \text{for } y < -1.$$

The overall bundle about (2.4) is 2D, which is obtained by the linearization:

$$(2.5) \quad f(y) = f_*(y) + Y(y) \implies [f_*(y)y]''' - \beta Y' y + \alpha Y = \dots = 0.$$

The linearly independent solutions are

$$(2.6) \quad Y(y) = y^m \implies (m-3)(m^2 + 9m - 20\alpha - 2) = 0.$$

Hence there exist two negative roots  $m_{\pm}$  composing a 2D stable manifold. Fortunately,  $f_*(y)$  is strictly negative for  $y < 0$  and attracts no orbits from the positive quarter-plane.

Secondly, the necessary behaviour at infinity of  $f(y)$  is:

$$(2.7) \quad f(y) = C_0 |y|^{\frac{3\alpha}{1+\alpha}} (1 + o(1)) \quad \text{as } y \rightarrow -\infty \quad \left(\frac{3\alpha}{1+\alpha} = \frac{\alpha}{\beta}\right),$$

where  $C_0 > 0$  is a constant, which can be arbitrarily changed by scaling (2.3). It is important for the future conclusions to derive the whole 3D bundle of solutions satisfying (2.7). As usual, this is done by the linearization as  $y \rightarrow -\infty$ :

$$(2.8) \quad \begin{aligned} f(y) &= f_0(y) + Y(y), \quad \text{where } f_0(y) = C_0 (-y)^{\frac{\alpha}{\beta}} \\ \implies C_0 [(-y)^{\frac{\alpha}{\beta}} Y]''' + \beta Y'(-y) + \alpha Y + \frac{1}{2} [f_0(y)]''' + \dots &= 0. \end{aligned}$$

According to classic WKBJ-type asymptotic techniques in ODE theory, we look for solutions of (2.8) in the exponential form with the following characteristic equation:

$$(2.9) \quad Y(y) \sim e^{a(-y)^{\gamma}}, \quad \gamma = 1 + \frac{1}{2} \left(1 - \frac{\alpha}{\beta}\right) > 1 \implies C_0 (\gamma a)^2 = -\beta, \quad a_{\pm} = i \frac{1}{\gamma} \sqrt{\frac{\beta}{C_0}}.$$

This gives the whole 3D bundle of the orbits (2.7): as  $y \rightarrow -\infty$ ,

$$(2.10) \quad f(y) = C_0 (-y)^{\frac{\alpha}{\beta}} + (-y)^{\delta} \left[ C_1 \sin\left(\frac{1}{\gamma} \sqrt{\frac{\beta}{C_0}} (-y)^{\gamma}\right) + C_2 \cos\left(\frac{1}{\gamma} \sqrt{\frac{\beta}{C_0}} (-y)^{\gamma}\right) \right] + \dots,$$

where  $C_{1,2} \in \mathbb{R}$ . The slow decaying factor  $(-y)^{\delta}$  in the double scale asymptotics (2.10) is not essential in what follows, so we do not specify the exponent  $\delta < 0$  therein.

The behaviour (2.9) and (2.10) give crucial for us asymptotics: due to (2.7), we have the gradient blow-up behaviour at a single point: for any fixed  $x < 0$ , as  $t \rightarrow 0^-$ , where  $y = x/(-t)^{\beta} \rightarrow -\infty$ ,

$$(2.11) \quad u_-(x, t) = (-t)^{\alpha} f(y) = (-t)^{\alpha} C_0 \left| \frac{x}{(-t)^{\beta}} \right|^{\frac{\alpha}{\beta}} (1 + o(1)) \rightarrow C_0 |x|^{\frac{3\alpha}{1+\alpha}},$$

uniformly on compact subsets, as required by (1.23).

The geometry of the above asymptotic bundles yields:

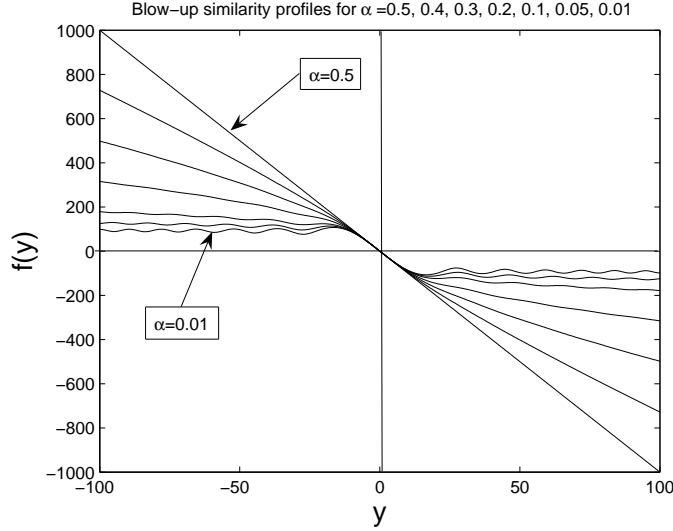


FIGURE 1. The odd blow-up similarity profiles  $f(y)$  in  $\mathbb{R}$  with  $f'(0) = -10$  and  $\alpha = 0.5, 0.4, 0.3, 0.2, 0.1$ , and  $0.01$ .

**Proposition 2.1.** *For any fixed  $A < 0$  in (2.2), the problem (1.21) admits the unique shock wave profile  $f(y)$ , which is an odd function and is strictly positive for  $y < 0$ .*

Uniqueness follows from the asymptotics (2.2) and scaling invariance (2.3). Global existence as infinite extension of the unique solution from  $y = 0^-$  follows from the structure and dimension of the bundles (2.10) (it is 3D, i.e., comprises the whole phase space of the equation in this quadrant). The 2D “bad” bundle, composed from the orbits (2.5), is not connected with the anti-symmetric ones. Concerning positivity, which is rather technical, see some details in [8, § 3,4].

Figure 1 shows a general view of similarity profiles  $f(y)$  for various values of the parameter  $\alpha \in [0.01, 0.5]$  with the fixed derivative

$$f'(0) = -10.$$

In Figure 2, we show enlarged oscillatory structure corresponding to (2.10) of some of the profiles closer to the origin. Figure 3 continues to explain the oscillatory behaviour (2.10) of solutions on different  $y$ -scales.

The next Figure 4 shows how the oscillatory features dramatically increase for negative  $\alpha$ . For  $\alpha = -0.099$ , which is very close to the critical value  $-\frac{1}{10}$  in (1.22), we observe a “saw-type” profile of maximally allowed oscillatory structure; see more details in [8, § 4].

Finally, we claim that, besides odd blow-up similarity profiles, there exist other not that symmetric, for which the conditions at the origin in (1.21) do not apply. This construction can be performed similar to that in [8, § 3,4]. The blow-up solutions constructed above are sufficient for our main purposes.



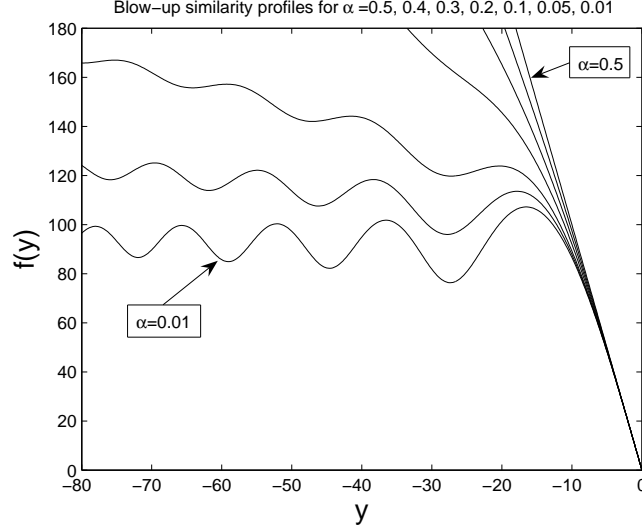


FIGURE 2. Enlarged oscillatory behaviour of  $f(y)$  from Figure 1 for  $y \in [-80, 0]$ .

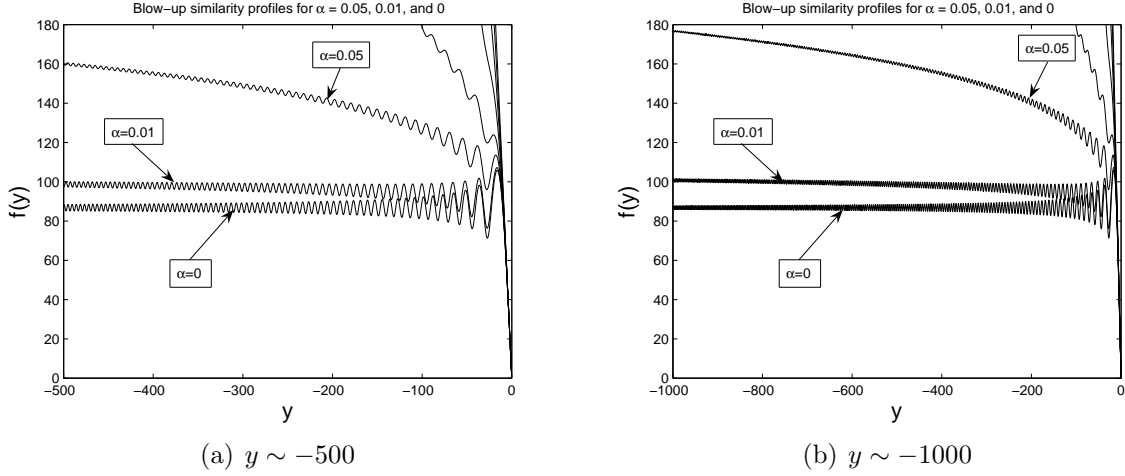


FIGURE 3. Enlarged oscillatory behaviour of  $f(y)$  from Figure 1 for  $y \in [-500, 0]$  (a) and  $y \in [-1000, 0]$  (b).

**2.1. On self-similar collapse of shocks.** It is curious that the similarity solutions (1.20) given by the ODE (1.21) can describe *collapse* as  $t \rightarrow 0^-$  of shocks. These are given by profiles  $f(y)$ , which instead of the anti-symmetry conditions in (1.21), satisfy

$$(2.12) \quad f(0) = f_0 > 0, \quad f'(0) = f_1 < 0, \quad \text{and} \quad f''(0) = f_2 \in \mathbb{R}.$$

Since the bundle at infinity (2.10) is 3D and hence exhausts all the trajectories there, shooting with the parameters (2.12) yields an orbit  $f(y; f_0, f_1, f_2)$ , which, for a wide range of  $f_{0,1,2}$ , has the behaviour (2.7) with a  $C_0 > 0$  and creates the data (2.11) as  $t \rightarrow 0^-$ .

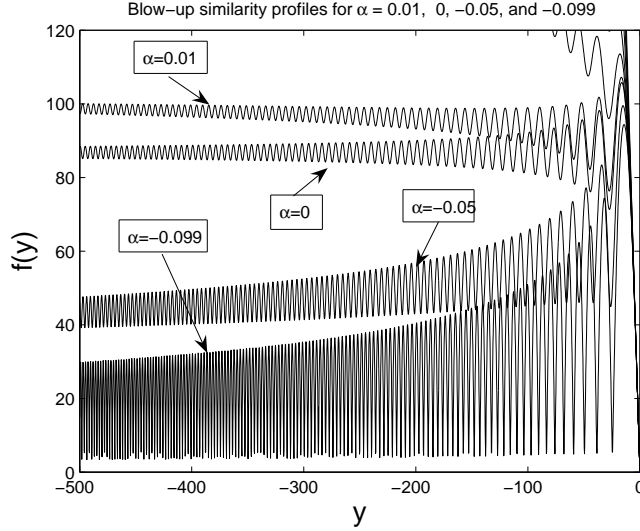


FIGURE 4. Oscillatory behaviour of  $f(y)$  for positive and negative  $\alpha$ .

In addition, for the corresponding similarity solution (1.20), the shock disappears since

$$(2.13) \quad [u_-(0, t)] = 2f_0(-t)^\alpha \rightarrow 0 \quad \text{as } t \rightarrow 0^-.$$

However, since the data at  $t = 0^-$  has a typical form (2.11), a new shock will be created as  $t \rightarrow 0^+$ , which we are going to explain in Section 3, where the parameterization such as in (2.12) for the ODE (1.25) will be used in greater detail.

**2.2. On smooth rarefaction waves.** These are global in time solutions of the form (1.24), with the ODE as in (1.25). Then taking  $F(y) \equiv -f(y)$ , one observes how the weakly singular initial data

$$(2.14) \quad -u_-(x, 0^-) \quad \text{given in (1.23)}$$

collapse into the smooth (even analytic) solution  $u_+(x, t)$  for  $t > 0$ . This is quite similar to the same phenomena as in [8] and [6], and we will not comment on this anymore, but indeed will use the global solutions (1.24), however in their discontinuous hypostasis.

### 3. NONUNIQUE SIMILARITY EXTENSIONS BEYOND BLOW-UP

**3.1. Nonuniqueness of similarity solutions.** As we mentioned, a discontinuous shock wave extension of blow-up solutions (1.20), (1.21) are performed by using the global ones (1.24), (1.25). Actually, this leads to watching a whole three-parametric family of solutions parameterized by their Cauchy values at the origin:

$$(3.1) \quad F(0) = F_0 > 0, \quad F'(0) = F_1 < 0, \quad \text{and} \quad F''(0) = F_2 \in \mathbb{R}.$$

The 3D phase space for the ODE in (1.25) has two clear stable “bad” bundles:

(I) positive solutions with “singular extinction” in finite  $y$ , where  $F(y) \rightarrow 0$  as  $y \rightarrow y_0^+ < 0$ . Indeed, this is an unavoidable singularity following from the degeneracy of the equations with the principal term  $FF'''$  leading to the singular potential  $\sim \frac{1}{F}$ ;

(II) positive solutions with the fast growth about the explicit solution:

$$(3.2) \quad F_*(y) = -\frac{y^3}{60} \rightarrow +\infty \quad \text{as } y \rightarrow -\infty.$$

The 2D stable bundle is similar to that in (2.5).

Both sets of such solutions are *open* by the standard continuous dependence of solutions of ODEs on parameters. The desired solutions are situated in between those two stable open bundles; cf. arguments in [8, § 3,4].

As usual, it is key to derive the whole 2D bundle of solutions satisfying (2.7). Similar to (2.8), we perform the standard linearization as  $y \rightarrow -\infty$  in (1.25):

$$(3.3) \quad \begin{aligned} f(y) &= F_0(y) + Y(y), \quad \text{where } F_0(y) = C_0(-y)^{\frac{\alpha}{\beta}} \\ \implies C_0[(-y)^{\frac{\alpha}{\beta}} Y]''' - \beta Y'(-y) - \alpha Y + \frac{1}{2} [F_0(y)]''' + \dots &= 0. \end{aligned}$$

The WKB method now leads to a different characteristic equation:

$$(3.4) \quad Y(y) \sim e^{a(-y)^\gamma}, \quad \gamma = 1 + \frac{1}{2} \left(1 - \frac{\alpha}{\beta}\right) > 1 \quad \implies \quad C_0(\gamma a)^2 = \beta, \quad a_\pm = \pm \frac{1}{\gamma} \sqrt{\frac{\beta}{C_0}},$$

so that the only admissible root is  $a_- < 0$ . This gives a 2D bundle of the orbits (2.7):

$$(3.5) \quad f(y) = C_0(-y)^{\frac{\alpha}{\beta}} + (-y)^\delta C_2 e^{a(-y)^\gamma} + \dots \quad \text{as } y \rightarrow -\infty, \quad C_1 \in \mathbb{R}.$$

As a result, we have the following:

**Proposition 3.1.** *Let  $\alpha \in (0, \frac{1}{2})$ . For any fixed  $F_0 > 0$  and  $F_1 < 0$  in (3.1), there exists a unique  $F_2 \in \mathbb{R}$  such that the problem (1.25) has a solution  $F_*(y)$  for some  $C_0 > 0$ .*

The proof is performed by shooting as in [8, 7] by using the stable bundles indicated in (i) and (ii) above. Such techniques are currently well-established for various higher-order ODEs. As a similar and more complicated example of a fourth-order ODE, we refer to the methods in [11], where by shooting technique existence and uniqueness of a positive solution of the radial bi-harmonic equations with source:

$$(3.6) \quad \Delta_r^2 u = u^p \quad \text{for } r = |x| > 0, \quad u(0) = 1, \quad u'(0) = u'''(0) = 0, \quad u(\infty) = 0,$$

was proved in the supercritical Sobolev range

$$p > p_{\text{Sob}} = \frac{N+4}{N-4}, \quad \text{where } N > 4.$$

Here, analogously, there exists a single shooting parameter being the second derivative at the origin  $F_2 = u''(0)$ ; the value  $F_0 = u(0) = 1$  is fixed by obvious scaling. Proving uniqueness of such a solution in [11] is not easy and lead to essential technicalities, which the attentive Reader can consult in case of necessity. Note that, instead of the global behaviour such as (3.2), the equation (3.6) admits the blow-up one governed by the principal operator

$$u^{(4)} + \dots = u^p \quad (u \rightarrow +\infty).$$

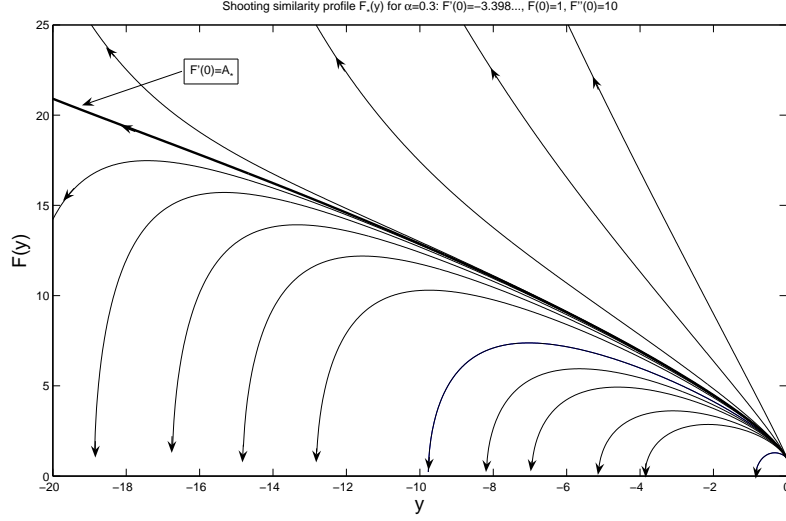


FIGURE 5. Shooting a proper solution  $F_*(y)$  of (1.25) for  $\alpha = 0.3$  and data  $F(0) = F_0 = 1$ ,  $F''(0) = F_2 = 10$ , with  $F_1 = F'(0) = A_* = -3.398\dots$

The solutions vanishing at finite point otherwise can be treated as in the family **(I)**.

Thus, as a result, we obtain a two-parametric family of solutions of (1.25) with an arbitrary fixed  $C_0 > 0$  parameterized by  $F_0 > 0$  and  $F_1 < 0$ . For a fixed constant  $C_0 > 0$  (uniquely given by the blowing up limit  $t \rightarrow 0^-$ ), the family is one-parametric:

$$(3.7) \quad \Phi_{C_0} = \{F_*(y; F_0), F_0 > 0\},$$

which for convenience we parameterize by the value  $F_0 = F_*(0)$  that measures the jump of the shock. The actual parameterization of the family (3.7) is not of importance. For instance, in Figure 5, we show shooting  $F_*(y)$  for  $\alpha = 0.3$ , which actually explains the strategy of proving of Proposition 3.1. In Figure 6, the same is done for  $\alpha = 0.1$  and smaller  $F(0) = 0.1$ ,  $F''(0) = 0$ . In Figure 7, again for  $\alpha = 0.1$ , we show a couple of more shooting with respect to parameters  $F_2$  (a) and  $F_1$  in (b).

**3.2. Remark: on nonexistence of solutions with  $F_0 = 0$ .** This is also a principal issue. Indeed, if a solution of (1.25) for  $F_0 = 0$ , i.e.,  $F(y)$  having the asymptotics near the origin similar to (2.2),

$$(3.8) \quad F(y) = Ay - \frac{1-2\alpha}{72} y^3 + \frac{(1-2\alpha)^2}{72^2} \frac{1}{A} y^5 + \dots \quad \text{as } y \rightarrow 0 \quad (A < 0),$$

would exist, then the similarity solution (1.24) would describe a smooth collapse of the initial singularity (1.23), and would actually mean an extra nonuniqueness in the problem. Fortunately, this is not the case and the single parameter  $A < 0$  in (3.8) (actually reducing to  $A = -1$  by scaling (2.3)) is not sufficient to shoot the necessary asymptotics as  $y \rightarrow -\infty$  given in the second line in (1.24), i.e., the corresponding asymptotic bundles as  $y \rightarrow 0^-$  and  $y \rightarrow -\infty$  are not overlapping. We do not prove this carefully (but indeed this can be done, since the “non-overlapping of bundles” is large enough), and, as a key illustration, present

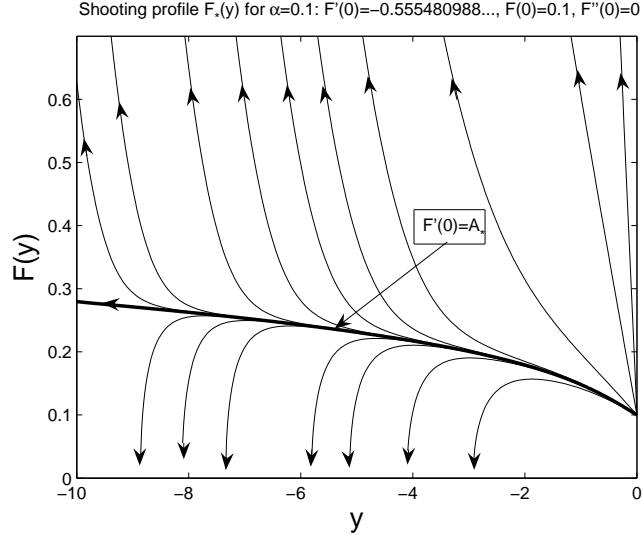
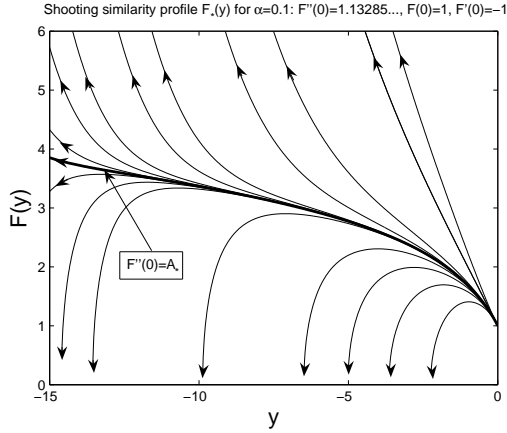
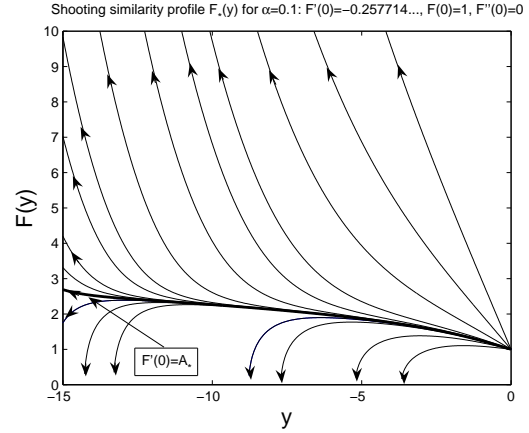


FIGURE 6. Shooting a proper solution  $F_*(y)$  of (1.25) for  $\alpha = 0.1$  and data  $F(0) = F_0 = 0.1$ ,  $F''(0) = F_2 = 0$ , with  $F_1 = F'(0) = A_* = -0.55548098\dots$



(a)  $F_0 = 1$ ,  $F_1 = -1$



(b)  $F_0 = 1$ ,  $F_2 = 0$

FIGURE 7. Shooting a proper solution  $F_*(y)$  of (1.25) for  $\alpha = 0.1$  and data  $F(0) = F_0 = 1$ ,  $F'(0) = F_1 = -1$ ,  $F''(0) = F_2 = A_* = 1.13285\dots$  (a) and  $F(0) = F_0 = 1$ ,  $F''(0) = F_2 = 0$ ,  $F'(0) = F_1 = A_* = -0.257714\dots$  (b).

Figure 8, where this nonexistence is carefully (with Tolerances about  $10^{-10}$ ) checked for  $\alpha = 0.1$  (a) and  $\alpha = 0.3$  (b).

However, for more complicated nonlinear PDEs, such as, e.g., [7]

$$u_{tt} = -(uu_x)_{xxx}, \quad \text{etc.},$$

where the resulting ODEs are higher order with multi-dimensional phase space, by some reasons, such solutions can be available, meaning that formation of shocks can be reversible and is not a unique option.

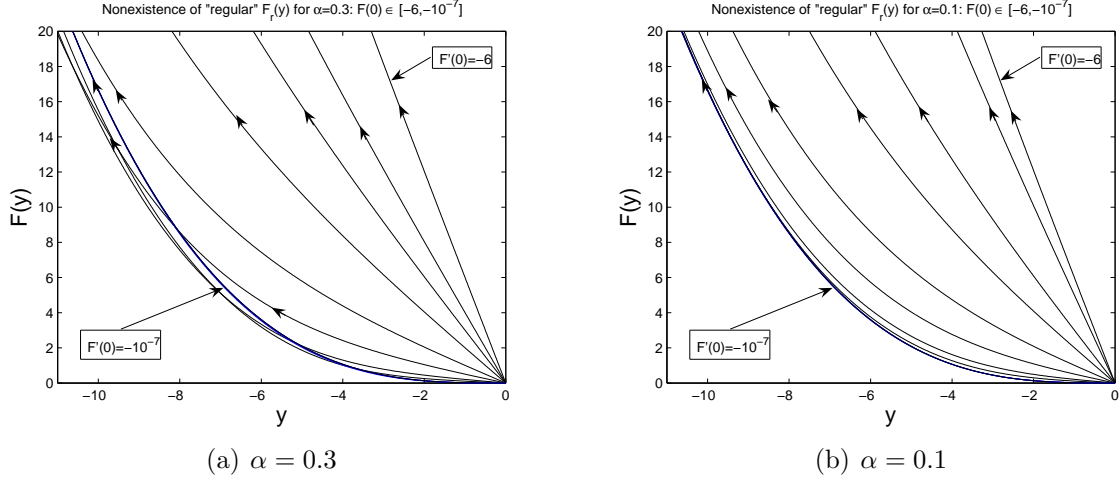


FIGURE 8. Towards nonexistence of a solution of (1.25), (3.8), with  $F_0 = 0$ , for  $\alpha = 0.3$  (a) and  $\alpha = 0.1$  (b).

**3.3. Further discussion around nonuniqueness and non-entropy issues.** Thus, we have explained the required non-uniqueness (1.28) of the solution of the Cauchy problem (1.1), (1.23): taking any profile  $F_* \in \Phi_{C_0}$  yields the self-similar continuation (1.24), with the following behaviour of the jump at  $x = 0$ :

$$(3.9) \quad -[u_+(x, t)]|_{x=0} \equiv -(u_+(0^+, t) - u_+(0^-, t)) = 2F_0 t^\alpha > 0 \quad \text{for } t > 0.$$

Rephrasing the result, let us emphasize that, in the similarity (i.e., an ODE) representation, it has a pure dimensional origin: the problem (1.25) has too many solutions. More precisely, for the given 3D shooting via parameters  $\{F_0, F_1, F_2\}$  in the 3D phase space of the ODE in (1.25), the dimension 1D of the non-suitable bundles in **(I)** and **(II)** is not sufficient to define a unique solution  $F_*(y)$ , up to scaling (2.3) as usual. In other words, the desired uniqueness could be achieved if

$$(3.10) \quad \text{the "bad" bundles in **(I)** and **(II)** are 2D in the 3D phase space}$$

(plus certain natural structural "transversality" as a genetic property). Hence, (3.10) is a geometric recognition of a possible uniqueness (and entropy that assumes extra hard work) extension of singular shock wave solutions.

Note that since these shocks are stationary, the corresponding Rankine–Hugoniot (the R–H) condition on the speed  $\lambda$  of the shock propagation:

$$(3.11) \quad \lambda = -\frac{[(uu_x)_x]}{[u]}|_{x=0} = 0$$

is valid by anti-symmetry. As usual, this condition is obtained by integration of the equation (1.1) in a small neighbourhood of the shock. Alternatively, (3.11) is obtained by approximating the solution via a travelling wave (TW)

$$(3.12) \quad u(x, t) = f(x - \lambda t) \implies -\lambda f' = (ff')'' \implies -\lambda f = (ff')' \implies \lambda = -\frac{[(ff')']}{[f]}|_{y=0},$$

which coincides with (3.11). Recall again that the R–H condition does not assume any novelty and is a corollary of integrating the PDE about the line of discontinuity.

Moreover, the R–H condition (3.11) also indicates another origin of nonuniqueness: a *symmetry breaking*. The point is that the solution for  $t > 0$  is not obliged to be an odd function of  $x$ , so we can define the self similar solution (1.24) for  $x < 0$  and  $x > 0$  using different triples of parameters  $\{F_0^\pm, F_1^\pm, F_2^\pm\}$ , and the only extra condition one needs is the R–H one:

$$(3.13) \quad \lambda = 0 \implies [(FF')'] = 0, \quad \text{i.e.,} \quad F_0^- F_2^- + (F_1^-)^2 = F_0^+ F_2^+ + (F_1^+)^2,$$

which can admit other solutions than the obvious anti-symmetric one

$$F_0^- = -F_0^+, \quad F_1^- = F_1^+, \quad \text{and} \quad F_2^- = -F_2^+.$$

Overall, in any case, we have at least a one-parameter family of solutions of the CP  $\Psi_{C_0} = \{u_+(x, t)\}$  for  $t > 0$  depending on arbitrary parameter  $F_0 > 0$ . This family *is not discrete* and hence does not reveal any particular solution  $u_+$ , exhibiting some special properties as being, say, maximal, minimal, or extremal in any sense. Under a natural assumption, this confirms the extra negative statement (1.29) on nonexistence of a sufficiently general entropy-like inequality, condition, and/or a procedure to detect a unique solution, as a “better” special one. At least, if even an “entropy-like” procedure would have been derived (somehow, by any hypothetical means), in view of the nonunique formation of shocks everywhere, the resulting solution would have never been relative to data in any metric, i.e., Hadamard’s well-posedness concept would have been violated anyway.

Indeed, different regularization of the NDE can lead to different solutions, i.e., by using a parabolic regularization:

$$(3.14) \quad u_\varepsilon : \quad u_t = (uu_x)_{xx} - \varepsilon u_{xxxx} \quad (\varepsilon > 0),$$

with the same data  $u_0(x)$ . Proving that the regularized sequence  $\{u_\varepsilon, \varepsilon > 0\}$  is a compact subset in some metrics is indeed a very difficult problem, which remains open for general data (see [5] for some details concerning such problems). Anyway and however, assuming that a suitable compactness of  $\{u_\varepsilon\}$  is already available, we do not think that a *proper unique solution*  $\bar{u}(x, t)$  can be obtained as the unconditional limit

$$(3.15) \quad \bar{u}(x, t) = \lim_{\varepsilon \rightarrow 0^+} u_\varepsilon(x, t).$$

More precisely, in view of the above nonuniqueness of post-blow-up similarity extensions, we believe that (3.15) has infinitely many partial limits and, along various subsequences  $\{\varepsilon_k\} \rightarrow 0^+$ , the corresponding sequences  $\{u_{\varepsilon_k}\}$  can converge to different solutions  $u_+$  with various values of rescaled shocks  $F_0 > 0$ . We justify this in an example below.

**Example.** We take  $\alpha = \frac{1}{5}$  and  $C_0 = 1$ , so that the blow-up initial data (1.23) are

$$(3.16) \quad u_0(x) = \sqrt{|x|} \quad \text{for} \quad x \leq 0,$$

with the odd extension for  $x > 0$ . Performing in (3.14) a natural scaling

$$(3.17) \quad u_\varepsilon(x, t) = \varepsilon^{\frac{1}{3}} v(y, \tau), \quad y = \frac{x}{\varepsilon^{2/3}}, \quad \tau = \frac{t}{\varepsilon^{5/3}}$$

deletes  $\varepsilon$  and yields the following uniformly parabolic problem:

$$(3.18) \quad v_\tau = (vv_y)_{yy} - v_{yyyy} \quad \text{in } \mathbb{R}_- \times \mathbb{R}_+, \quad v_0(y) = \sqrt{|y|} \quad \text{in } \mathbb{R}_-,$$

with the anti-symmetry conditions  $v = v_{yy} = 0$  at  $y = 0$ . It is not a great deal, by using classic parabolic theory, to prove global existence of a unique classical solution of the CP (3.18), though it takes some efforts. But the main issue is not existence/uniqueness. Indeed, according to (3.17), the study of the convergence (3.15) as  $\varepsilon \rightarrow 0^+$  reduces to delicate asymptotic behaviour of  $v(y, \tau)$  simultaneously as  $\tau \rightarrow +\infty$  and  $y \rightarrow -\infty$ , which represents a very difficult problem. However, the above detected nonuniqueness of the similarity extensions makes such an open asymptotic problem excessive and not that necessary (quite fortunately it seems). Note that studying the behaviour of the hypothetical proper limit (3.15) at the point of discontinuity  $x = 0$  assumes detecting the limit of  $\{u_\varepsilon(x_k, t)\}$  ( $t > 0$  small is fixed) for rather arbitrary sequences  $\{x_k\} \rightarrow 0^-$  as  $\varepsilon \rightarrow 0$ . We then claim that such a limit essentially depends of the choice of  $\{\varepsilon_k\} \rightarrow 0^+$ , i.e., the limit of the sequence

$$(3.19) \quad u_{\varepsilon_k}(x_k, t) \equiv \varepsilon_k^{\frac{1}{3}} v\left(\frac{x_k}{\varepsilon_k^{2/3}}, \frac{t}{\varepsilon_k^{5/3}}\right) \quad \text{as } k \rightarrow \infty$$

is very much  $\{x_k\}$ - and  $\{\varepsilon_k\}$ -dependent.

We also expect that using other types of regularization, e.g., by classic Bubnov–Galerkin methods of finite-dimensional approximations on suitable functional Riesz bases (see e.g., strong applications in Lions’ classic monograph [13]), which obviously give globally existing solutions, will also lead to similar nonuniqueness issues in the limit<sup>2</sup>, so these are unavoidable difficulties of modern PDE theory.

In other words, we then expect that, for a number of higher-order<sup>3</sup> nonlinear PDEs with singularities such as blow-up, extinction, or shock wave formation, fully consistent (see above) general entropy-like procedures to reveal a unique solution cannot be available in principle. As we have shown, this is just prohibited by a sufficient dimension of the phase space (depending on the spatial order  $\geq 3$  of the PDE), which allows to shoot a continuous family of solutions beyond singularity, whose family does not have any isolated and/or boundary points.

**3.4. On uniqueness for the FBP setting.** Evidently, the uniqueness can be restored if an extra condition at the shocks is assumed, which poses an FBP for (1.1). For instance, following Figure 6 and 7(b), this happens if we fix

$$(3.20) \quad F_2 = F'''(0) = 0 \quad \implies \quad u_{xx}(0^\pm, t) = 0.$$

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<sup>2</sup>Therefore, the issue of the *Galerkin uniqueness*, which can be treated as a simplest idea for an entropy-like construction, becomes author’s unrealizable dream.

<sup>3</sup>This is a principle issue: for general second-order nonlinear parabolic equations with blow-up and obeying the Maximum Principle, there always exists a unique *proper minimal solution*, which does not depend on the type of monotone regularization of the equation; see [10, § 2] or [4, Ch. 6].



Indeed, the uniqueness (in the present self-similar setting) is restored if the set (3.7) contains a unique such profile, i.e.,

$$(3.21) \quad \Phi_{C_0} \cap \{F''_*(0) = 0\} = \{\hat{F}_*(y), \text{ with a fixed } \hat{F}_0 > 0\}.$$

Then this fixes the unique “shock divergence” (3.9) with  $F_0 = \hat{F}_0$  beyond the singularity. In a PDE setting, existence-uniqueness of a solution of the FBP (1.1), (3.20) is a difficult problem, which, for some simple geometric configuration of shocks, can be solved by traditional FBP methods, such as von Mises transformations and others. In general, mathematical difficulties can be extremely challenging.

A more general free-boundary condition on shock lines can be predicted from the structure (1.24) of (generic) similarity continuation beyond blow-up. One should only take into account that parameters  $\alpha$  and  $\beta$  cannot enter such conditions, since these essentially depend on *a priori* unknown blow-up “initial data”  $u(x, 0^-)$ . Since at the shock at  $x = 0$ , by (1.24)

$$(3.22) \quad u_+ = t^\alpha F_0, \quad (u_+)_x = t^{\alpha-\beta} F_1, \quad \text{and} \quad (u_+)_{xx} = t^{\alpha-2\beta} F_2,$$

it is easy to reconstruct a general FBP condition at shocks, which suits for arbitrary  $\alpha$ :

$$(3.23) \quad uu_{xx} = \kappa(u_x)^2, \quad \text{where} \quad \kappa \in \mathbb{R}.$$

Since both sides are of order  $t^{2\alpha-2\beta}$ , this condition well corresponds to any of similarity formation of shocks (what happens for other configurations, is another delicate story). Then (3.20) is obtained for  $\kappa = 0$ . Formally,  $\kappa = \infty$  yields  $F_1 = 0$ , i.e., a kind of

$$\text{a “Neumann” FBP : } u_x = 0 \quad \text{at shocks.}$$

Of course, it is necessary to check, for which  $\kappa$  the condition

$$(3.24) \quad F_0 F_2 = \kappa(F_1)^2$$

yields a unique profile  $F_* \in \Phi_{C_0}$ , and this occurs for arbitrary  $\alpha$ . As customary, posing necessary free-boundary conditions is an applied physical issue, though checking and predicting the well-posedness of the FBPs occurred is indeed a mathematical problem.

#### 4. FINAL REMARKS: ON ORIGIN OF UNIQUENESS FOR THE 1D EULER EQUATION

In this connection, it is interesting to trace out similar origins of the uniqueness in the CP for the Euler equation (1.9). Its obvious advantage is that it is solved via characteristics and the solutions are given by an algebraic relation:

$$(4.1) \quad \frac{dt}{1} = \frac{dx}{u} \implies x - ut = \text{const.} \implies u(x, t) = u_0(x - u(x, t)t).$$

Let us assume the same initial data (1.23), so that for  $x \leq 0$  we have the equation

$$(4.2) \quad u = C_0(ut - x)^{\frac{\alpha}{\beta}}, \quad \text{where now, dimensionally, } \beta = 1 + \alpha.$$

Obviously, setting here  $x = 0$ , corresponding to the permanent position of this stationary shock, yields the *unique* value of the solution at the shock:

$$(4.3) \quad u = C_0(ut)^{\frac{\alpha}{\beta}} \implies u(0^-, t) = C_0^{\frac{\beta}{\beta-\alpha}} t^{\frac{\alpha}{\beta-\alpha}} \equiv C_0^\beta t^\alpha \quad \text{for } \beta = 1 + \alpha.$$

Or, analogously, the same uniqueness is guaranteed by the fact that, for the first-order ODEs corresponding to rescaling of (1.9), the phase space is also one-dimensional, that allows a unique (and very simple) matching of two bundles. The corresponding similarity solutions for  $t < 0$  and  $t > 0$  respectively are given by:

$$(4.4) \quad \begin{aligned} u_-(x, t) &= (-t)^\alpha f(y), \quad y = \frac{x}{(-t)^\beta} \implies f f' + \beta f' y - \alpha f = 0, \\ u_+(x, t) &= t^\alpha F(y), \quad y = \frac{x}{t^\beta} \implies F F' - \beta F' y + \alpha F = 0, \end{aligned}$$

where  $\beta = 1 + \alpha$ , and then as in (4.3) we have to have

$$(4.5) \quad u_-(0^-, t) = C_0^\beta t^\alpha \quad \text{for } t > 0, \text{ i.e., } F(0) = C_0^\beta.$$

Of course, the ODEs in (4.4) are explicitly integrated. As usual, the last one for the post-blowing up behaviour, is responsible for a unique continuation beyond blow-up. Integrating it yields the unique solution  $F_*(y)$  defined by the algebraic equation

$$(4.6) \quad F' = -\frac{\alpha F}{F - \beta y}, \quad F = -yP, \quad yP' = -\frac{P(P+1)}{P+\beta} \implies \frac{F^\beta(y)}{[F(y) + |y|]^\alpha} = C_0^\beta \quad \text{for } y \leq 0,$$

where  $C_0$  is the constant in (4.2) (a similar procedure applies to the blow-up profile  $f$ ). It then follows from (4.6) that the rescaled shock jump is also uniquely determined:

$$(4.7) \quad F_0 = F_*(0) = C_0^\beta,$$

and indeed this precisely gives the time behaviour (4.3) and (4.5) of the shock value.

Note that, unlike calculus in (3.13), a symmetry breaking is obviously impossible here. Indeed, the R–H condition uniquely implies the symmetry:

$$(4.8) \quad \lambda = \frac{[F^2]}{2[F]} = \frac{(F_0^+)^2 - (F_0^-)^2}{2(F_0^+ - F_0^-)} \equiv \frac{F_0^- + F_0^+}{2} = 0 \implies F_0^+ = -F_0^-.$$

It is not an exaggeration to say that this unique micro-structural extension of generic blow-up singularities at any point eventually reflects a true dimensional and ODE origin of those classic entropy theories constructed for conservation laws in  $\mathbb{R}$  by Oleinik in the 1950s and by Kruzhkov in 1960s for equations in  $\mathbb{R}^N$ . Indeed, if, for the simplest ODE similarity problem for (1.9), the matching would be nonunique and non-discrete (plus something else), these very influential theories would have never been appeared and created. We thus claim that this nonuniqueness and non-entropy are precisely the case for the Cauchy problem for the NDE (1.1) and can be expected for a number of other nonlinear dispersion (and not only those) higher-order equations from PDE theory of the twenty first century.

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Shooting similarity profile  $F_*(y)$  for  $\alpha=0.3$ ;  $F'(0)=A_*=-0.94108\dots$ ,  $F(0)=1$ ,  $F''(0)=0$

